If possible let α be the phase difference between E and H, i.e

$$E = E_0 e^{i(kx-\omega t)}$$

$$H = H_0 e^{i(kx-\omega t+\alpha)}$$

The fields E and H have to satisfy the Maxwell's equation

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

therefore

$$\frac{\partial E_z}{\partial x} = \frac{\partial B_y}{\partial t}; \quad \frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}$$

(Therefore $\frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ of E is zero since E travels in x-direction)

From the second eq^n

$$ik E_{0y} e^{i(kx-\omega t)} = i\omega\mu H_{0z} e^{i(kx-\omega t+\alpha)}$$

Taking the real parts

$$k E_{0y} \cos(kx-\omega t) = \omega\mu H_{0z} \cos(kx-\omega t+\alpha)$$

This is true for all x and t.

The only value of α which satisfies this condition is zero. Therefore E and H are in phase. The ratio of amplitudes of E and H is given by

$$\frac{E_0}{H_0} = \frac{\mu\omega}{k} = Z_0 \quad \text{—(1)}$$

The ratio Z_0 has dimension of impedance and is called the intrinsic impedance of the medium.

Thus, a monochromatic plane-wave always has

E and H in phase and with the ratio of amplitudes at any instant at any point given by

$$\frac{E}{H} = \frac{\mu\omega}{k} = \mu v \quad \text{--- (2)}$$

for a plane wave in free space

$$Z_0 = \mu_0 v_0 = 376.7\,\Omega \quad \text{--- (3)}$$

for the propagation of the wave in any arbitrary direction, we have

$$E(r,t) = E_0 e^{i(k.r - \omega t)} \quad \text{--- (4)}$$

$$H(r,t) = H_0 e^{i(k.r - \omega t)} \quad \text{--- (5)}$$

$E_0, H_0 \rightarrow$ vectors constant in time

$k = \hat{e}_k |k|$ propagation vector

$\hat{e}_k \rightarrow$ unit vector in the direction of propagation.

Since E and H are real, we are interested only in the real part of eqn (4) and (5)

Now E and H obtained as above must also satisfy Maxwell's equation.

Putting eqn (4) in maxwell (i) eqn

$$\nabla . E_0 e^{i(k.r - \omega t)} = 0 \quad (\because \rho = 0)$$

$$\Rightarrow k.E = 0 \quad \text{--- (6)}$$

Similarly from Maxwell (ii) eqn

$$k.H = 0 \quad \text{--- (7)}$$

This shows that both E and H are perpendicular to the propagation vector k.

Such a wave is called <u>transverse wave</u>.

Electromagnetic plane waves are wholly transverse in character.

Now substitute eq^n (4) in Maxwell (iii) eq^s

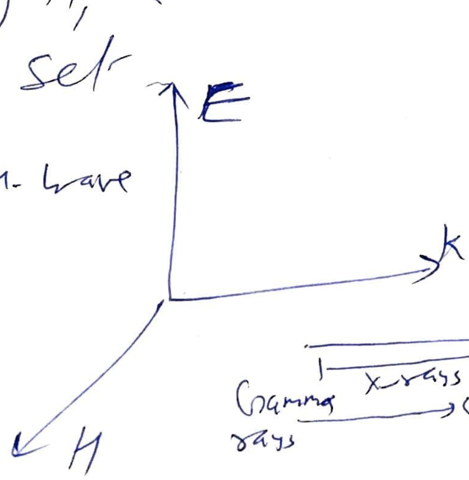
$$\nabla \times E = \nabla \times E_0 e^{i(k\cdot r - \omega t)}$$

$$= \nabla e^{i(k\cdot r - \omega t)} \times E_0$$

$$= ik \times E = -\frac{\partial B}{\partial t} = -\mu \frac{\partial H}{\partial t} = i\omega\mu H$$

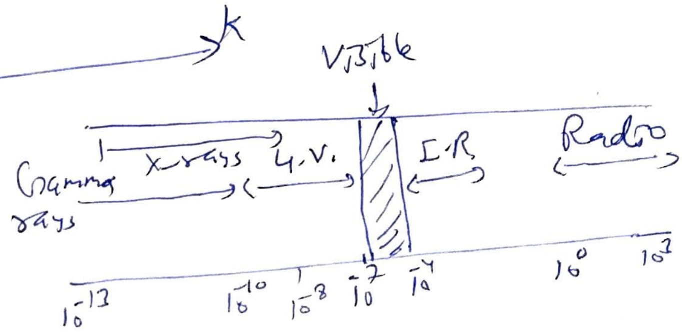
$$\Rightarrow \quad k \times E = \omega \mu H \quad \text{--- (8)}$$

Hence, H is perpendicular both to k and E. E and H, which relate to the propagation of the wave, in addition to being perpendicular to the direction of propagation are also perpendicular to one another. The vector $E \times H$ points along the direction of propagation. The vectors E, H, k constitute a right hand orthogonal set.



The velocity of E-M wave in free space is given by

$$v_0 = \frac{1}{(\varepsilon_0 \mu_0)^{1/2}}$$



$\varepsilon_0 = 8.85 \times 10^{-12}$, $\mu_0 = 4\pi \times 10^{-7}$

$$v_0 = 2.99784 \times 10^{8} \text{ m/s} \quad \text{--- (9)}$$

In medium $\rightarrow$ $v = \frac{1}{(\varepsilon\mu)^{1/2}} = \frac{1}{(\varepsilon_0 \varepsilon_r \mu_0 \mu_r)^{1/2}} = \frac{v_0}{(\varepsilon_r \mu_r)^{1/2}}$ --- (10)

which is less than v_0. For a light in non-dispersive medium $v = \frac{c}{n}$, $n \rightarrow$ refractive index of the medium, $n = (\varepsilon_r \mu_r)^{1/2}$

Field Energy and field Momentum

The general expressions for the electrostatic and magnetostatic field energies are:

$$W_E = \frac{1}{2}\int_V (E \cdot D)\, d\tau \quad \text{and} \quad W_M = \frac{1}{2}\int_V (B \cdot H)\, d\tau \quad \text{--- (1)}$$

We find the expression for the electromagnetic energy in time-dependent situations.

The force on a moving charge q is given by

$$F = q\,(E + v \times B) \quad \text{--- (2)}$$

The rate at which the work is done on this charge is

$$F \cdot v = q\,(E + v \times B)\cdot v = qE \cdot v \quad \text{--- (3)}$$

Magnetic field → does no work as it is perpendicular to the velocity of the charge.

If there exists a continuous distribution of charge, the total rate at which the work is done in a given volume is

$$\int \rho\,(E \cdot v)\, d\tau = \int (E \cdot j)\, d\tau \quad \text{--- (4)}$$

$$\int (E \cdot j)\, d\tau = \int \left[E \cdot \left(\nabla \times H - \frac{\partial D}{\partial t}\right)\right] d\tau \quad \text{--- (5)}$$

Now $E \cdot (\nabla \times H) = \nabla \cdot (H \times E) + H \cdot (\nabla \times E)$

$$= \nabla \cdot (H \times E) - H \cdot \frac{\partial B}{\partial t}$$

$$\Rightarrow \int (E \cdot j)\, d\tau = \int \left[\nabla \cdot (H \times E) - E \cdot \frac{\partial D}{\partial t} - H \cdot \frac{\partial B}{\partial t}\right] d\tau$$

$$\Rightarrow \int (E\cdot j)\, d\tau = \int [\nabla\cdot(H\times E)]\, d\tau - \int \left(E\cdot\frac{\partial D}{\partial t} + H\cdot\frac{\partial B}{\partial t}\right) d\tau \quad (6)$$

Using Divergence theorem to transform the first integral, we have

$$\int (E\cdot j)\, d\tau = \int_S (H\times E)\cdot\hat{e}_n\, ds - \int_V \left(E\cdot\frac{\partial D}{\partial t} + H\cdot\frac{\partial B}{\partial t}\right) d\tau$$

$S \rightarrow$ Surface boundary of the volume V. Therefore

$$-\int_V \left(E\cdot\frac{\partial D}{\partial t} + H\cdot\frac{\partial B}{\partial t}\right) d\tau = \int (E\cdot j)\, d\tau + \int_S (E\times H)\cdot\hat{e}_n\, ds \quad (7)$$

If ε and $\mu \rightarrow$ are assumed to be constant, we have for the linear media

$$\int_V \left(E\cdot\frac{\partial D}{\partial t} + H\cdot\frac{\partial B}{\partial t}\right) d\tau = \frac{\partial}{\partial t}\int_V \frac{1}{2}[E\cdot D + B\cdot H]\, d\tau$$

Therefore

$$-\frac{\partial}{\partial t}\int_V \frac{1}{2}[E\cdot D + B\cdot H]\, d\tau = \int (E\cdot j)\, d\tau + \int (E\times H)\cdot\hat{e}_n\, ds \quad (8)$$

Electromagnetic field $\rightarrow$ consists of electric and magnetic fields, we may assume that the sum of the energies given in eq^n (1) represents the total electromagnetic energy. We may take

$\varepsilon_M = \frac{1}{2}[E\cdot D + B\cdot H] \rightarrow$ electromagnetic energy density.

L.H.S. of eq^n (8) $\rightarrow$ rate at which the energy stored in the electromagnetic field diminishes.

1st term on R.H.S. $\rightarrow$ work done by the field forces on the charges contained in the volume.

Last term is eq^n (8) :— energy that flows out of the boundary per unit time.

The vector [E × H] which gives the rate at which the energy flows across unit area of the boundary is called the Poynting vector → represented by N.

The eq^n (8) may be written in differential form as

$$\frac{d\mathcal{E}_M}{dt} + \operatorname{div} N = -E \cdot j \quad \text{——} (9)$$

If the medium has zero conductivity

$j = \sigma E = 0$, Hence

$$\frac{d\mathcal{E}_M}{dt} + \operatorname{div} N = 0 \quad \text{——} (10)$$

This eq^n has exactly the same form as the eq^n of continuity. $\left[\nabla \cdot j + \frac{\partial \rho}{\partial t} \right]$ ⇒ flux of e.m. energy poynting vector $N = E \times H$

Eq^n (8) represents the law of conservation of energy. It states that the decrease of electromagnetic energy per unit time in a certain volume V is equal to the work done by the field forces per unit time plus the flux flow outward per unit time ⟶ This is known as

Poynting Theorem

$E \times H$ → has dimension of $\frac{\text{energy}}{\text{area} \times \text{time}}$